

ALGEBRAIC VECTOR BUNDLES ON PUNCTURED AFFINE SPACES AND SMOOTH QUADRICS

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ABSTRACT. There is a canonical \mathbb{A}^n -fibration from the $(2n+1)$ -dimensional smooth quadric Y_n to the $(n+1)$ -dimensional punctured affine space X_n . For each rank at least n , we construct examples of non-isomorphic algebraic vector bundles on the punctured affine space with isomorphic pullbacks to the smooth quadric. Moreover, we construct continuous families of arbitrarily large dimensional pairwise non-isomorphic rank n bundles on X_n with trivial pullbacks to the smooth quadric.

1. INTRODUCTION

Given a smooth affine variety X , denote by $\mathcal{V}_n(X)$ the isomorphism classes of rank n algebraic vector bundles on X . Morel proved that ¹ (cf. [Mo]),

$$\mathcal{V}_n(X) = [X, BGL_n]_{\mathbb{A}^1}.$$

Here, BGL_n is the simplicial classifying space of GL_n (cf. [MV]) and $[\cdot, \cdot]_{\mathbb{A}^1}$ denotes the equivalence classes of maps in the \mathbb{A}^1 -homotopy category.

The above theorem might make one hope that some form of homotopy invariance holds for the functor $\mathcal{V}_n(X)$ beyond the affine case. Indeed, by the Jouanolou-Thomason homotopy lemma (cf. [Wei], Proposition 4.4), given a smooth scheme X admitting an ample family of line bundles (e.g., a quasi-projective variety), there exists a smooth affine scheme Y and a Zariski locally trivial smooth morphism $f : Y \rightarrow X$ with fibers isomorphic to affine spaces (f is called an affine vector bundle torsor). In particular, this morphism is an \mathbb{A}^1 -weak equivalence. Thus the above naive hope would reduce the study of vector bundles on such schemes to the case of affine varieties. Unfortunately, Theorem 1.2 in [AD] shows that this is false. In [AD], Asok and Doran constructed continuous families of \mathbb{A}^1 -contractible smooth varieties, which admit continuous moduli of non-trivial algebraic vector bundles.

Given $n \geq 1$, denote by

$$Y_n = SL_{n+1} / SL_n$$

and

$$X_n = \mathbb{C}^{n+1} \setminus \{0\},$$

where $SL_n = SL(n, \mathbb{C})$. Let

$$p : Y_n \rightarrow X_n$$

be the projection which maps an $(n+1) \times (n+1)$ matrix to its first row (cf. Section 4). Then p is a fiber bundle with fibers isomorphic to \mathbb{C}^n . Namely, Y_n is an *affine vector bundle torsor* over X_n . From the point of view of the \mathbb{A}^1 -homotopy theory, p is an \mathbb{A}^1 -homotopy weak equivalence. Since the algebraic K -theory is representable in the \mathbb{A}^1 -homotopy category (cf. Theorem 2.3.1 in [Vo]), it follows that p induces an isomorphism of algebraic K -groups. In particular, we have $K_0(Y_n) = K_0(X_n)$. Therefore, it is natural to ask the following question:

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¹with the rank 2 case completed by Moser

Question 1.1. *Does there exist a pair of non-isomorphic algebraic vector bundles (E_1, E_2) on X_n such that their pull-backs to Y_n satisfy $p^*(E_1) \cong p^*(E_2)$?*

In this chapter, we give an affirmative answer to Question 1.1. In fact, given that $\text{rank } r \geq n$, we show that there exist non-trivial bundles on X_n of rank r satisfying that their pull-backs to Y_n are trivial (cf. Theorem 5.5). Moreover, we construct continuous families of pairwise non-isomorphic algebraic bundles on X_n of rank n of arbitrarily large dimension such that their pull-backs to Y_n are trivial (cf. Theorem 5.8). We sketch the proof as follows. First we define an invariant e for algebraic vector bundles on \mathbb{P}^n . Then we use it to study the pull-backs to X_n and Y_n of bundles on \mathbb{P}^n . Given an algebraic vector bundle E on \mathbb{P}^n , $e(E)$ is a non-negative integer. Most of our studies are about the bundles E with $e(E) = 1$. We also note that Asok and Fasel have made progress reducing Morel's abstract solution to concrete computation of \mathbb{A}^1 -homotopy groups of algebraic spheres (cf. [AF]).

Notation and conventions. Let

$$\pi : X_n \longrightarrow \mathbb{P}^n$$

be the natural projection and let

$$\rho = \pi \circ p.$$

Let $\mathcal{O}_{\mathbb{P}^n}(k)$ be the line bundle on \mathbb{P}^n corresponding to the divisor kH , where H is a hyperplane divisor. Given an algebraic vector bundle E on \mathbb{P}^n , let $E(k) = E \otimes \mathcal{O}_{\mathbb{P}^n}(k)$ and $H^i(E)$ denote the i -th (sheaf) cohomology group of E .

Throughout, we assume that varieties in this chapter are over the complex field. However, all results except Swan-Towber's and Swan's theorems and Theorem 5.4 (2) hold true for varieties over any algebraically closed field.

2. VECTOR BUNDLES ON AFFINE SPACES, PROJECTIVE SPACES AND PUNCTURED AFFINE SPACES

The study of algebraic vector bundles on an affine scheme $\text{Spec } A$ is equivalent to the study of projective A -modules. In the 1950s, Serre asked whether algebraic vector bundles on the affine space \mathbb{C}^n are trivial. This was addressed by Quillen and Suslin independently in 1976. They proved that: over a field k , any projective $k[x_1, \dots, x_n]$ -module of finite rank is trivial.

In the 1950s Grothendieck showed that, any algebraic vector bundle on the projective line \mathbb{P}^1 is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(k)$. In [Sc2], Schwarzenberger introduced *almost decomposable bundles*. An algebraic vector bundle on $X = \mathbb{P}^2$ is called almost decomposable if $\dim H^0(X, \text{End } E) > 1$, where $\text{End } E = E \otimes E^*$ is the endomorphism bundle of E . Schwarzenberger classified all the almost decomposable bundles on \mathbb{P}^2 of rank 2 in [Sc]. In [Sc2], Schwarzenberger showed that, any rank 2 bundle E on $X = \mathbb{P}^2$ is of the form $E = f_*M$ where M is a line bundle on a non-singular Y and $f : Y \longrightarrow X$ is a ramified double covering. In the case that $Y = Q_2 = \mathbb{P}^1 \times \mathbb{P}^1$ or $Y = V_2$, a blowing-up of \mathbb{P}^2 with seven distinct points as base points, Schwarzenberger studied the rank 2 bundles $E = f_*M$ intensively. Since the 1970s, the study of algebraic vector bundles on projective spaces has focused on stable (and semi-stable) vector bundles and their moduli spaces. In the 1970s Barth and Hulek proved that, the moduli space $M_{\mathbb{P}^2}(c_1, c_2)$ of stable vector bundles E on $X = \mathbb{P}^2$ with fixed Chern classes $(c_1(E), c_2(E)) = (c_1, c_2)$ is an irreducible, rational and smooth variety (cf. [B] and [Hu]). The books [OSS] and [HL] give an excellent introduction to the study of vector bundles on projective spaces.

In [Ho], Horrocks studied algebraic vector bundles on the punctured spectrum $Y = \text{Spec } A - \{m\}$ for a regular integral ring A and an maximal ideal m of it. He defined Φ -equivalence for algebraic vector bundles on Y and showed that ([Ho], Theorem 7.4),

a bundle \mathcal{E} on Y is determined up to Φ -equivalence by the cohomology modules $H^i(\mathcal{E})$ ($0 < i < \dim A - 1$) and the extensions $b^i(\mathcal{E}) \in \text{Ext}_A^2(H^{i+1}(\mathcal{E}), H^i(\mathcal{E}))$ ($0 < i < \dim A - 2$). As an application, Horrocks got the following beautiful criterion:

Theorem 2.1 (Horrocks). *An algebraic vector bundle E of rank r on \mathbb{P}^n is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$ if and only if $H^i(E(k)) = 0$ for any $0 < i < n$ and any $k \in \mathbb{Z}$, where $E(k) = E \otimes \mathcal{O}_{\mathbb{P}^n}(k)$.*

3. UNIMODULAR AND COMPLETABLE ROWS

In this section, we give a brief review of the theory of unimodular rows. The readers could refer [L] for more details. Given a commutative ring R , let $\text{GL}_n(R)$ be the group of $n \times n$ matrices with entries in R . A row vector $\vec{a} = (a_0, a_1, \dots, a_n)$ is called a *unimodular* row if $Ra_0 + Ra_1 + \dots + Ra_n = R$. Two unimodular rows \vec{a}^1, \vec{a}^2 are called equivalent, if $\vec{a}^2 = \vec{a}^1 g$ for some $g \in \text{GL}_{n+1}(R)$. We denote by $\vec{a}^1 \sim \vec{a}^2$, if \vec{a}^1 and \vec{a}^2 are equivalent. A unimodular row \vec{a} is called *completable* if $\vec{a} \sim (1, 0, \dots, 0)$.

Given a unimodular row \vec{a} , there is a surjective homomorphism

$$R^{n+1} \longrightarrow R, \quad \sum_{0 \leq i \leq n} x_i e_i \mapsto \sum_{0 \leq i \leq n} a_i x_i,$$

where $\{e_0, \dots, e_n\}$ is the standard basis of R^{n+1} . Let $P_{\vec{a}}$ be the kernel of this surjective homomorphism, which is a projective R -module of rank n . It is known that $P_{\vec{a}^1} \cong P_{\vec{a}^2}$ if and only if $\vec{a}^1 \sim \vec{a}^2$. In particular, $P_{\vec{a}}$ is trivial if and only if \vec{a} is completable.

We define the ring $R = \mathbb{C}[x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n] / \langle x_0 y_0 + x_1 y_1 + \dots + x_n y_n - 1 \rangle$, and a unimodular row of the form $(f_0, \dots, f_n) = (x_0^{a_0}, \dots, x_n^{a_n})$, where a_0, \dots, a_n are non-negative integers. We restate the celebrated $n!$ Theorem of Suslin and its converse due to Swan-Towber.

Theorem 3.1. *Let $(f_0, \dots, f_n) = (x_0^{a_0}, \dots, x_n^{a_n})$, where $a_i \in \mathbb{Z}_{\geq 1}$.*

- (1) (Suslin, [L]) *If $n! \mid \prod a_i$, then $(f_0, f_1, \dots, f_n) \in R^{n+1}$ is completable.*
- (2) (Swan-Towber, [ST]) *If $n! \nmid \prod a_i$, then $(f_0, f_1, \dots, f_n) \in R^{n+1}$ is not completable.*

We remark that Suslin's theorem holds on any field.

Given a unimodular row (f_0, \dots, f_n) with f_i a homogeneous polynomial of degree a_i , the following generalization of Theorem 3.1 states that:

Theorem 3.2 ([Ku] and [Sw]). *Let $\{f_0, \dots, f_n\}$ be homogeneous polynomials of x_0, \dots, x_n satisfying that*

$$\text{rad}(f_0, f_1, \dots, f_n) = (x_0, x_1, \dots, x_n).$$

- 1), (Mohan Kumar, [Ku]) *If $n! \mid \prod \deg f_i$, then $(f_0, f_1, \dots, f_n) \in R^{n+1}$ is completable.*
- 2), (Swan, [Sw]) *If $n! \nmid \prod \deg f_i$, then $(f_0, f_1, \dots, f_n) \in R^{n+1}$ is not completable.*

Mohan Kumar's theorem holds for any algebraic closed field.

4. INVARIANT e AND SOME EXAMPLES

Let $Y_n = \text{SL}_{n+1} / \text{SL}_n$ and $X_n = \mathbb{C}^{n+1} \setminus \{0\}$, and let \mathbb{P}^n be n -dimensional projective space over \mathbb{C} . Here SL_n is included in SL_{n+1} in the following way

$$\text{SL}_n = \{\text{diag}\{1, B\} \in \text{SL}_{n+1} \mid B \in \text{SL}_n\}.$$

Let $p : Y_n \rightarrow X_n$ be the map $p([A]) = (a_{00}, \dots, a_{0,n})$ (the first row) for any

$$A = (a_{i,j})_{0 \leq i,j \leq n} \in \text{SL}_{n+1},$$

where

$$[A] = \mathrm{SL}_n A \in \mathrm{SL}_{n+1} / \mathrm{SL}_n = Y_n.$$

Let $\pi : X_n \rightarrow \mathbb{P}^n$ be the natural projection, and

$$\rho = \pi \circ p : Y_n \rightarrow \mathbb{P}^n.$$

For $A = (a_{i,j})_{0 \leq i,j \leq n} \in \mathrm{SL}_{n+1}$, let $x_i = a_{0,i}$ and

$$y_i = (-1)^i \det A_i$$

where A_i is the $n \times n$ sub-matrix of A with the first row (indexed by 0) and the $(i+1)$ -th column (indexed by i) deleted. Then we have the isomorphism

$$\mathrm{SL}_{n+1} / \mathrm{SL}_n \cong \{(x_0, \dots, x_n, y_0, \dots, y_n) \in \mathbb{C}^{2n+2} : \sum_{0 \leq i \leq n} x_i y_i = 1\}.$$

The last equality implies that Y_n is isomorphic to the $(2n+1)$ -dimensional smooth quadric.

Let

$$R = R[Y_n] = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_n] / \langle \sum_{0 \leq i \leq n} x_i y_i - 1 \rangle$$

and $S = R[X_n] = \mathbb{C}[x_0, x_1, \dots, x_n]$. Let $m = (x_0, x_1, \dots, x_n)$ be the maximal ideal corresponding to the point $0 = \mathrm{Spec} S - X_n$.

Given a vector bundle E of rank r on \mathbb{P}^n and $i \in \mathbb{Z}$, define

$$M_i(E) = \bigoplus_{k \in \mathbb{Z}} H^i(E(k)),$$

where $E(k) = E \otimes \mathcal{O}_{\mathbb{P}^n}(k)$. Let $M_E = M_0(E)$ and $M'_E = M_E / m M_E$.

Proposition 4.1. *Given an algebraic vector bundle E of rank r on \mathbb{P}^n ,*

- (1) *if $i < 0$ or $i > n$, then $M_i(E) = 0$.*
- (2) *For any i with $0 < i < n$, $M_i(E)$ is of finite dimension.*
- (3) *M_E is a finitely generated $S = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(k)$ module.*
- (4) *M'_E is of finite dimension.*

Proof. For (1), if $i < 0$, it is clear that $H^i(E(k)) = 0$ for any integer k . Thus $M_i(E) = 0$. If $i > n$, for any $k \in \mathbb{Z}$, by the Serre duality $H^i(E(k)) = (H^{n-i}(E(-n-1-k)))^* = 0$. Hence $M_i(E) = 0$.

For (2), since $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n}(1)$ is very ample, given an $i > 0$ there exists a $k_i \in \mathbb{Z}$ such that $H^i(E(k)) = 0$ for $k \geq k_i$ (cf. [H], Theorem 5.2). By the Serre duality, for any $i < n$, $H^i(E(k)) = (H^{n-i}(E^*(-n-1-k)))^*$. Thus there exists a $k'_i \in \mathbb{Z}$ such that $H^i(E(k)) = 0$ for $k \leq k'_i$. Since each $H^i(E(k))$ is of finite-dimension, thus $M_i(E)$ is of finite dimension.

For (3), it follows from [HL], Lemma 1.7.2 and [H], Theorem 5.2 that, M_E is a finitely generated $S = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(k)$ module.

Finally, (4) follows from (3). □

Definition 4.2. *Given an algebraic vector bundle E of rank r on \mathbb{P}^n , define $e(E) + r$ to be the minimal number of generators of the S -module M_E .*

Lemma 4.3. *Given an algebraic vector bundle E of rank r on \mathbb{P}^n , we have $e = e(E) = \dim_{\mathbb{C}}(M'_E) - r$. Moreover, there exists an exact sequence*

$$0 \longrightarrow E' \longrightarrow \bigoplus_{1 \leq i \leq e+r} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E \longrightarrow 0$$

for some integers a_1, \dots, a_{e+r} and a vector bundle E' of rank e . In particular we have $e(E) \geq 0$.

Proof. If the elements x_1, x_2, \dots, x_{e+r} generate the S -module M_E , then they generate the $S/m = \mathbb{C}$ vector space $M'_E = M_E/mM_E$. Thus $\dim_{\mathbb{C}}(M'_E) \leq e(E) + r$. On the other hand, by Proposition 4.1, M_E is a finitely generated S -module. Thus there exists $k_0 \in \mathbb{Z}$ such that $H^0(E(k)) = 0$ for $k < k_0$. Let x_1, \dots, x_t be homogeneous elements of M_E such that,

$$[x_1] = x_1 + mM_E, \dots, [x_t] = x_t + mM_E$$

is a basis of M'_E . From the condition of $H^0(E(k)) = 0$ for $k < k_0$, one can show that x_1, \dots, x_t generate the S -module M_E . Hence $e(E) + r \leq \dim_{\mathbb{C}}(M'_E)$. Therefore $e(E) = \dim_{\mathbb{C}}(M'_E) - r$.

By the argument above, there exists a system of homogeneous generators

$$\{x_i \in H^0(E(-a_i)) \mid 1 \leq i \leq e+r\}$$

of the S -module M_E . For each i , the element $x_i \in H^0(E(-a_i))$ gives a map

$$]\mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E.$$

Summing them up, we get a map of vector bundles

$$\psi : \bigoplus_{1 \leq i \leq e+r} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E.$$

By Theorem 5.17 in [H], ψ is a surjective map of vector bundles. Let E' be the kernel of ψ . Then we get an exact sequence

$$0 \longrightarrow E' \longrightarrow \bigoplus_{1 \leq i \leq e+r} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E \longrightarrow 0.$$

It is clear that $\text{rank } E' = (e(E) + r) - \text{rank } E = e(E)$. Therefore $e(E) \geq 0$. \square

Corollary 4.4. *Given an algebraic vector bundle E of rank r on \mathbb{P}^n , $e(E) \geq 0$.*

Given two algebraic vector bundles E_1, E_2 of finite rank on \mathbb{P}^n , $e(E_1 \oplus E_2) = e(E_1) + e(E_2)$.

Proof. The first statement follows from the second statement in Lemma 4.3. The second statement follows from the first statement in Lemma 4.3. \square

Proposition 4.5. *Given an algebraic vector bundle E on \mathbb{P}^n with $e(E) = 1$, there exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(b) \longrightarrow \bigoplus_{1 \leq i \leq r+1} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E \longrightarrow 0$$

for some integers b, a_1, \dots, a_r .

Proof. This follows from Lemma 4.3. Note that, since $\text{rank } E' = e(E) = 1$, we have $E' \cong \mathcal{O}_{\mathbb{P}^n}(b)$ for some integer b . \square

Theorem 4.6. *Given an algebraic vector bundle E of rank r on \mathbb{P}^n , the following conditions are equivalent to each other:*

- (1) $e(E) = 0$.
- (2) E is a direct sum of line bundles.
- (3) π^*E is a trivial bundle on X .
- (4) $H^i(\pi^*E) = 0$ for any $1 \leq i \leq n-1$.
- (5) $H^i(E(k)) = 0$ for any $1 \leq i \leq n-1$ and any $k \in \mathbb{Z}$,

Proof. Lemma 4.3 implies that (1) \Leftrightarrow (2). It follows from Theorem 2.3.1 in [OSS] that (5) \Leftrightarrow (2). It is obvious that (2) \Rightarrow (3) and (3) \Rightarrow (4).

To show (4) \Leftrightarrow (5), we consider

$$\pi : X_n = \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n,$$

which is a fibre bundle with fibers all isomorphic to $\mathbb{C}^* = \mathbb{C} - \{0\}$. A straightforward calculation shows that $\pi_*(\pi^*E) \cong \bigoplus_{k \in \mathbb{Z}} E(k)$. Since π is an affine morphism, by the Leray spectral sequence we get

$$H^i(\pi^*E) = H^i(\pi_*(\pi^*E)) = \bigoplus_{k \in \mathbb{Z}} H^i(E(k)).$$

From this equality, we get (4) \Leftrightarrow (5). \square

Proposition 4.7. *Let E be a vector bundle of rank r on \mathbb{P}^n with $M_i(E) = 0$ for $1 \leq i \leq n-2$. Then we have an exact sequence*

$$0 \longrightarrow \bigoplus_{1 \leq i \leq e} \mathcal{O}_{\mathbb{P}^n}(b_i) \longrightarrow \bigoplus_{1 \leq i \leq r+e} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E \longrightarrow 0$$

for some integers $b_1, \dots, b_e, a_1, \dots, a_{e+r}$.

Proof. By Lemma 4.3, we have an exact sequence

$$0 \longrightarrow E' \longrightarrow \bigoplus_{1 \leq i \leq e+r} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E \longrightarrow 0$$

for some integers a_1, \dots, a_{e+r} and a vector bundle E' of rank e . Let

$$E'' = \bigoplus_{1 \leq i \leq e+r} \mathcal{O}_{\mathbb{P}^n}(a_i)$$

and denote by $\psi : E'' \longrightarrow E$ the second map. By the proof of Lemma 4.3, the map $H^0(E''(k)) \longrightarrow H^0(E(k))$ is surjective for $k \in \mathbb{Z}$.

From the short exact sequence $0 \longrightarrow E' \longrightarrow E'' \longrightarrow E \longrightarrow 0$, we get

$$H^{i-1}(E''(k)) \longrightarrow H^{i-1}(E(k)) \longrightarrow H^i(E'(k)) \longrightarrow H^i(E''(k)).$$

For $2 \leq i \leq n-1$, we have $H^{i-1}(E''(k)) = H^i(E''(k)) = 0$, since $E''(k)$ is a direct sum of line bundles. By the assumption of $H^{i-1}(E(k)) = 0$, we get $H^i(E'(k)) = 0$. For $i = 1$, $H^1(E''(k)) = 0$, since $E''(k)$ is a direct sum of line bundles and the map $H^0(E''(k)) \longrightarrow H^0(E(k))$ is surjective by the proof of Lemma 4.3. Therefore, $H^1(E'(k)) = 0$.

By these, we get $H^i(E(k)) = 0$ for $1 \leq i \leq n-1$ and $k \in \mathbb{Z}$. By Theorem 4.6, E' is a direct sum of line bundles. \square

Theorem 4.8. *Let E be a vector bundle of rank r on \mathbb{P}^n with $e(E) > 0$. Then,*

- (1) $\pi^*E \oplus (\mathcal{O}_{X_n})^t$ is not trivial for $t \geq 0$.
- (2) If $M_i(E) = 0$ for $1 \leq i \leq n-2$, then $\rho^*E \oplus (\mathcal{O}_{Y_n})^t$ is trivial for $t \geq e$.

Proof. For (1), note that $\pi^*E \oplus (\mathcal{O}_{X_n})^t = \pi^*(E \oplus (\mathcal{O}_{\mathbb{P}^n})^t)$ and $e(E \oplus (\mathcal{O}_{\mathbb{P}^n})^t) = e(E) > 0$. Hence $\pi^*E \oplus (\mathcal{O}_{X_n})^t$ is not trivial by Theorem 4.6.

For (2), by Proposition 4.7 we have the following exact sequence

$$0 \longrightarrow \bigoplus_{1 \leq i \leq e} \mathcal{O}_{\mathbb{P}^n}(b_i) \longrightarrow \bigoplus_{1 \leq i \leq r+e} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow E \longrightarrow 0.$$

Since $\rho : Y_n \rightarrow \mathbb{P}^n$ is an affine morphism, pulling back to Y_n , we get an exact sequence

$$0 \longrightarrow (\mathcal{O}_{Y_n})^e \longrightarrow (\mathcal{O}_{Y_n})^{e+r} \longrightarrow \rho^*E \longrightarrow 0.$$

Since Y_n is affine, the above short exact sequence of vector bundles splits. Thus

$$\rho^*E \oplus (\mathcal{O}_{Y_n})^t \cong (\rho^*E \oplus (\mathcal{O}_{Y_n})^e) \oplus (\mathcal{O}_{Y_n})^{t-e} \cong (\mathcal{O}_{Y_n})^{e+r} \oplus (\mathcal{O}_{Y_n})^{t-e} \cong (\mathcal{O}_{Y_n})^{t+r}$$

is trivial, if $t \geq e$. \square

Note that, the assumption of $M_i(E) = 0$ for $1 \leq i \leq n-2$ holds true if $n = 2$.

Example 4.9 (An explicit and simple example). *Let $G = \mathrm{SL}_3$,*

$$P = \left\{ \begin{pmatrix} \lambda & \alpha^t \\ 0_2 & B \end{pmatrix} \mid B \in \mathrm{GL}_2, \lambda \det B = 1, \alpha \in \mathbb{C}^2 \right\},$$

$$P' = \left\{ \begin{pmatrix} 1 & \alpha^t \\ 0_2 & B \end{pmatrix} \mid B \in \mathrm{SL}_2, \alpha \in \mathbb{C}^2 \right\}$$

and

$$H = \left\{ \begin{pmatrix} 1 & 0_2^t \\ 0_2 & B \end{pmatrix} \mid B \in \mathrm{SL}_2 \right\}.$$

Let \mathfrak{g} , \mathfrak{p} , \mathfrak{p}' and \mathfrak{h} be their Lie algebras. We have the following identifications

$$Y_2 = G/H,$$

$$X_2 = G/P'$$

and

$$\mathbb{P}^2 = G/P.$$

Let

$$E_1 = G \times_P (\mathfrak{g}/\mathfrak{p}')$$

and

$$E_2 = G \times_P (\mathfrak{g}/\mathfrak{p} \oplus \mathfrak{p}/\mathfrak{p}').$$

They are algebraic vector bundles on $\mathbb{P}^2 = G/P$ of rank 3. Then $\pi^*(E_1)$ and $\pi^*(E_2)$ are non-isomorphic on X_2 . However, their pull-backs to Y_2 are isomorphic.

Proof. $\pi^*(E_1) = G \times_{P'} (\mathfrak{g}/\mathfrak{p}')$ is the tangent bundle of $G/P' \cong \mathbb{C}^3 \setminus \{0\}$. The tangent bundle of $\mathbb{C}^3 \setminus \{0\}$ is clearly trivial, thus $\pi^*(E_1) \cong (\mathcal{O}_{X_2})^3$. By the Euler sequence (cf. Page 6 in [OSS])

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow (\mathcal{O}_{\mathbb{P}^n})^{n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0,$$

we get $e(T_{\mathbb{P}^n}) = 1$. In the case of $n = 2$, we get $e(E_2) = e(T_{\mathbb{P}^2}) = 1$. Hence $\pi^*(E_2)$ is not trivial. Therefore $\pi^*(E_1) \not\cong \pi^*(E_2)$.

Since H is semisimple, $\mathfrak{g}/\mathfrak{p}' \cong \mathfrak{g}/\mathfrak{p} \oplus \mathfrak{p}/\mathfrak{p}'$ as H -modules. Hence $\rho^*(E_1) \cong \rho^*(E_2)$. \square

5. BUNDLES WITH $e = 1$ AND MORE EXAMPLES

Let E be a rank r vector bundle on \mathbb{P}^n with $e(E) = 1$. By Proposition 4.5, we have an exact sequence

$$0 \longrightarrow E' = \mathcal{O}_{\mathbb{P}^n}(b) \longrightarrow E'' = \bigoplus_{0 \leq i \leq r} \mathcal{O}_{\mathbb{P}^n}(b_i) \longrightarrow E \longrightarrow 0$$

such that the map $H^0(E''(k)) \longrightarrow H^0(E(k))$ is surjective for $k \in \mathbb{Z}$. One can show that the bundle E determines the numbers b, b_0, \dots, b_r . Moreover we have $b < \min\{b_0, b_1, \dots, b_r\}$. The reason is that: if $b = \min\{b_0, b_1, \dots, b_r\}$, then E is a direct sum of line bundles, which contradicts to the fact that $e(E) = 1$. Moreover, the maps $\phi : E' \longrightarrow E''$ and $\psi : E'' \longrightarrow E$ are determined by E up to linear changes of bases.

Let $a_i = b_i - b > 0$ and $f_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(a_i)) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(b), \mathcal{O}_{\mathbb{P}^n}(b_i))$. Then for an algebraic vector bundle E on \mathbb{P}^n with rank r and $e(E) = 1$, we associate E with the integer b ,

positive integers a_0, \dots, a_r and homogeneous polynomials $f_0, f_1, \dots, f_r \in \mathbb{C}[x_0, x_1, \dots, x_n]$ of degree a_0, a_1, \dots, a_r , respectively.

Proposition 5.1. *Given $a_i > 0$ and $f_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(a_i))$, $0 \leq i \leq r$, the following conditions are equivalent to each other:*

- (1) *The map $\phi = (f_0, \dots, f_r) : \mathcal{O}_{\mathbb{P}^n} \longrightarrow \bigoplus_{0 \leq i \leq r} \mathcal{O}_{\mathbb{P}^n}(a_i)$ is an injective map of algebraic vector bundles.*
- (2) *The zero locus of (f_0, \dots, f_r) in \mathbb{P}^n is empty.*
- (3) *$\text{rad}(f_0, f_1, \dots, f_r) = (x_0, x_1, \dots, x_n)$ in $\mathbb{C}[x_0, \dots, x_n]$.*

In the case that these conditions are satisfied, we have that $r \geq n$ and $E = \text{Coker } \phi$ is a bundle of rank r with $e(E) = 1$.

Proof. Since the locus of the points in \mathbb{P}^n where ϕ is not injective is equal to the zero locus of (f_0, \dots, f_r) , (1) is equivalent to (2). It is well-known that (2) \Leftrightarrow (3).

In the case that (1) – (3) hold, $r \geq n$ since the Krull dimension of $\mathbb{C}[x_0, \dots, x_n]$ is $n + 1$. It is clear that $\text{Coker } \phi$ is an algebraic vector bundle of rank r .

Write $E' = \mathcal{O}_{\mathbb{P}^n}$, $E'' = \bigoplus_{0 \leq i \leq r} \mathcal{O}_{\mathbb{P}^n}(a_i)$ and $E = \text{Coker } \phi$. By the short exact sequence $0 \longrightarrow E' \longrightarrow E'' \longrightarrow E \longrightarrow 0$ and $H^1(E'(k)) = 0$ for $k \in \mathbb{Z}$, we get a short exact sequence

$$0 \longrightarrow H^0(E'(k)) \longrightarrow H^0(E''(k)) \longrightarrow H^0(E(k)) \longrightarrow 0.$$

Since $a_i > 0$ for $0 \leq i \leq r$, $\bigoplus_{k < 0} H^0(E''(k))$ generates $\bigoplus_{k \in \mathbb{Z}} H^0(E''(k))$ as an S -module. Thus $\bigoplus_{k < 0} H^0(E(k))$ generates $\bigoplus_{k \in \mathbb{Z}} H^0(E(k))$. As $H^0(E'(k)) = 0$ for $k < 0$, $H^0(E''(k)) \longrightarrow H^0(E(k))$ is an isomorphism for $k < 0$. Hence $\bigoplus_{k \in \mathbb{Z}} H^0(E(k))$ and $\bigoplus_{k \in \mathbb{Z}} H^0(E''(k))$ have the same minimal number of generators (cf. Lemma 4.3 and its proof), which is equal to $r + 1$. Therefore $e(E) = (r + 1) - r = 1$. \square

Lemma 5.2. *If $\text{rank } E = r \geq n + 1$ and $e(E) = 1$, then there exist $r + 1$ homogeneous generators f_0, f_1, \dots, f_r of M_E such that the zero locus of*

$$(f_0, f_1, \dots, f_{i_0-1}, f_{i_0+1}, \dots, f_{r-1}, f_r)$$

is still empty for some i_0 .

Proof. Suppose we can not find a system of $r + 1$ generators such that the zero locus of

$$[(f_0, f_1, \dots, f_{i_0-1}, f_{i_0+1}, \dots, f_{r-1}, f_r)]$$

is still empty for some i_0 . We prove by induction that, for any system of $r + 1$ generators (f_0, f_1, \dots, f_r) of M_E , any $1 \leq k \leq n + 1$ and any subset $I \subset \{0, 1, \dots, r\}$ of cardinality k , the zero locus of $\{f_i \mid i \in \{0, 1, \dots, r\} - I\}$ has dimension $k - 1$.

For $k = 1$, by assumption the zero locus Z_i of $(f_0, f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{r-1}, f_r)$ is non-empty for each i . However, $Z_i \cap Z(f_i) = \emptyset$ by the unimodularity condition. Thus $\dim Z_i \leq 0$. Hence $\dim Z_i = 0$. This proves the assertion in the case of $k = 1$.

For $1 \leq k \leq n$, suppose the assertion holds for $1, 2, \dots, k$. We prove that the zero locus $Z_{0,1,\dots,k}$ of (f_{k+1}, \dots, f_r) has dimension k . By the induction hypothesis, $Z_{0,1,\dots,k} \cap Z(f_k) = Z_{0,1,\dots,k-1}$ has dimension $k - 1$. Since $Z(f_k)$ is a hypersurface, we get $\dim Z_{0,1,\dots,k} \leq k$ by the Projective Dimension Theorem (cf. [H], Theorem 7.2). On the other hand, we have $\dim Z_{0,1,\dots,k} \geq \dim Z_{0,1,\dots,k-1} = k - 1$. Thus $\dim Z_{0,1,\dots,k} = k - 1$ or k . If $\dim Z_{0,1,\dots,k} \neq k$, then $\dim Z_{0,1,\dots,k} = k - 1$. Hence $Z_{0,1,\dots,k}$ is a $(k - 1)$ -dimensional closed subset of \mathbb{P}^n . Let C_1, \dots, C_s be all $(k - 1)$ -dimensional irreducible components of $Z_{0,1,\dots,k}$. Without loss of generality we may assume that $\deg f_{k-1} \leq \deg f_k$. Let $l = \deg f_k - \deg f_{k-1} \geq 0$. Choose a linear function g nonvanishing on each C_j , $1 \leq j \leq s$. Since $Z_{0,1,\dots,k} \cap Z(f_k) \cap Z(f_{k-1}) = Z_{0,1,\dots,k-2}$ has dimension $k - 2 < k - 1$, we have $C_j \not\subset Z(f_k)$ or $C_j \not\subset Z(f_{k-1})$ for each $1 \leq j \leq s$. Thus at least one of f_k and f_{k-1} does not vanish on C_j . Since g does not

vanish on C_j as well and C_j is irreducible, at least one of f_k and $g^l f_{k-1}$ does not vanish on C_j . Write S_j for the set of points $[\lambda_0, \lambda_1] \in \mathbb{P}^1$ such that $\lambda_0 f_k + \lambda_1 f_{k-1} g^l$ vanishes on C_j . Hence each S_j has at most one point. Choose a point $[1, \lambda_1] \in \mathbb{P}^1$ not lying in $\cup_{1 \leq k \leq s} S_j$. Let

$$f'_k = f_k + \lambda_1 f_{k-1} g^l.$$

Then f'_k does not vanish on each C_j , $1 \leq j \leq s$. That is to say, $C_j \not\subset Z(f'_k)$. Hence

$$\dim(C_j \cap Z(f'_k)) < \dim C_j = k - 1,$$

since C_j is irreducible. Moreover, we get

$$\dim(Z_{0,1,\dots,k} \cap Z(f'_k)) \leq k - 2.$$

However,

$$f_0, \dots, f_{k-1}, f'_k = f_k + \lambda_1 f_{k-1} g^l, f_{k+1}, \dots, f_r$$

is also a system of generators of M_E . Hence

$$\dim(Z_{0,1,\dots,k} \cap Z(f'_k)) \leq k - 2$$

contradicts the induction hypothesis. Therefore $Z_{0,1,\dots,k}$ has dimension k .

Similarly, we can show that the zero locus Z_{i_0, i_1, \dots, i_k} of

$$\{f_i \mid i \in \{0, 1, \dots, r\} - \{i_0, i_1, \dots, i_k\}\}$$

has dimension k for each set of indices $0 \leq i_0 < \dots < i_k \leq r$. In this way we conclude the induction step.

Let $k = n + 1 \leq r$. Then the zero locus of f_0, \dots, f_{r-n-1} has dimension $k - 1 \geq n$. It is impossible as it is a proper closed subset of \mathbb{P}^n . Therefore we reach the conclusion of the Lemma. \square

Theorem 5.3. *Let E be an algebraic vector bundle on \mathbb{P}^n with $e(E) = 1$. If $\text{rank } E = r \geq n + 1$, then $\rho^* E$ is trivial.*

Proof. By Propositions 4.5 and 5.1, E is determined by the integer b , $r+1$ positive integers a_0, \dots, a_r ($a_0 \leq a_1 \leq \dots \leq a_{r-1} \leq a_r$) and homogeneous polynomials f_0, \dots, f_r with f_i of degree a_i , $0 \leq i \leq r$. These homogeneous polynomials f_0, \dots, f_r arise from a choice of generators of $M_E = \bigoplus_{k \in \mathbb{Z}} H^0(E(k))$, and they are not uniquely determined by E . For example, if

$$f'_i = f_i + \sum_{0 \leq j \leq i-1} g_{i,j} f_j, \quad 0 \leq i \leq r,$$

with $g_{i,j}$ a homogeneous polynomial of degree $a_i - a_j$, then f'_0, \dots, f'_r and f_0, \dots, f_r define isomorphic vector bundles.

By Lemma 5.2, we can choose f_0, f_1, \dots, f_r such that the zero locus of

$$(f_0, f_1, \dots, f_{i_0-1}, f_{i_0+1}, \dots, f_{r-1}, f_r)$$

is empty for some $0 \leq i_0 \leq r$. We have an exact sequence

$$0 \longrightarrow E' = \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi} E'' = \bigoplus_{0 \leq i \leq r} \mathcal{O}_{\mathbb{P}^n}(a_i) \xrightarrow{\psi} E(-b) \longrightarrow 0.$$

Write

$$E_1 = \bigoplus_{0 \leq i \leq r, i \neq i_0} \mathcal{O}_{\mathbb{P}^n}(a_i)$$

and

$$E_2 = \mathcal{O}_{\mathbb{P}^n}(a_{i_0}).$$

Let $p : E'' \longrightarrow E_1$ be the natural projection and $\phi' = p \circ \phi$. Since the zero locus of

$$[(f_0, f_1, \dots, f_{i_0-1}, f_{i_0+1}, \dots, f_{r-1}, f_r)]$$

is empty, $\phi' = p \circ \phi$ is an injective map of vector bundles. Write $E_3 = \text{Coker } \phi'$. We have an exact sequence

$$0 \longrightarrow E_2 \longrightarrow E(-b) \longrightarrow E_3 \longrightarrow 0.$$

Thus, we get

$$\begin{aligned} & \rho^* E \\ \cong & \rho^*(E_2) \oplus \rho^*(E_3) \text{ (since } Y \text{ is affine)} \\ \cong & \mathcal{O}_Y \oplus \rho^*(E_3) \text{ (since } E_2 \text{ is a line bundle)} \\ \cong & (\mathcal{O}_Y)^r \text{ (since } e(E_3) = 1). \end{aligned}$$

□

Theorem 5.4. *Given an algebraic vector bundle E on $P = \mathbb{P}^n$ of rank n and with $e(E) = 1$, let a_0, a_1, \dots, a_n be the degrees of homogeneous polynomials defining E .*

- (1) *If $n! \mid a_0 a_1 \cdots a_n$, then $\rho^* E$ is trivial.*
- (2) *If $n! \nmid a_0 a_1 \cdots a_n$, then $\rho^* E$ is not trivial.*

Proof. This follows from Proposition 4.5, Proposition 5.1 and Theorem 3.2. □

Theorem 5.5. *Given an integer $r \geq n$, there exists an algebraic vector bundle E on $P = \mathbb{P}^n$ of rank r with $\pi^* E$ non-trivial and $\rho^* E$ trivial.*

Proof. This follows from Theorem 4.6, Proposition 5.1 and Theorem 5.4. □

Lemma 5.6. *Let E be an algebraic vector bundle on \mathbb{P}^n with $e(E) = 1$. If $\text{rank } E = n$, then E is indecomposable.*

Proof. Suppose that E is decomposable, i.e., $E = E_1 \oplus E_2$ where E_1 and E_2 are non-zero algebraic vector bundles on \mathbb{P}^n . Since $1 = e(E) = e(E_1) + e(E_2)$, we have $e(E_1) = 1$ or $e(E_2) = 1$. We may assume that $e(E_1) = 1$. By Proposition 5.1, we have $\text{rank } E_1 \geq n$. Thus $\text{rank } E = \text{rank } E_1 + \text{rank } E_2 \geq n + 1$, which contradicts to the fact that $\text{rank } E = n$. □

Theorem 5.7 ([Ho], Proposition 9.5). *Let E_1, E_2 be two indecomposable algebraic vector bundles on \mathbb{P}^n of the same rank. If $\pi^*(E_1) \cong \pi^*(E_2)$, then $E_2 \cong E_1(k)$ for some $k \in \mathbb{Z}$.*

Theorem 5.8. *There exist continuous families of arbitrarily large dimension of rank n algebraic vector bundles on X_n satisfying that the bundles in each family are pairwise non-isomorphic and their pull-backs to Y_n are trivial bundles.*

Proof. Choose a positive integer a such that

$$\left(\prod_{p \leq n} p \right) \mid a,$$

where \prod runs over all primes $p \leq n$. Let $a_0 = a_1 = \cdots = a_n = a$. Then

$$n! \mid a^{n+1} = a_0 a_1 \cdots a_n.$$

Denote by V the set of homogeneous polynomials of degree a . It is a linear vector space. Write $\text{Gr}_{n+1}(V)$ for the Grassmannian variety of $(n+1)$ -dimensional subspaces of V . Define

$$Z = \{ \text{span}_{\mathbb{C}} \{f_0, \dots, f_n\} \in \text{Gr}_{n+1}(V) : \text{rad}(f_0, \dots, f_n) \neq (x_0, \dots, x_n) \}$$

and

$$Z' = \{ ([W], [(z_0, \dots, z_n)]) \in \text{Gr}_{n+1}(V) \times \mathbb{P}^n : g(a_0, \dots, a_n) = 0, \forall g \in W \}.$$

Then Z is the image of Z' under the projection of $\text{Gr}_{n+1}(V) \times \mathbb{P}^n$ to its first component. Since Z' is Zariski closed and the above projection map is proper, Z is a Zariski closed subset of $\text{Gr}_{n+1}(V)$. On the other hand, it is clear that Z is a proper subset of $\text{Gr}_{n+1}(V)$. Thus its complement is an open dense subset of $\text{Gr}_{n+1}(V)$.

For a sequence $f = (f_0, f_1, \dots, f_n)$ such that

$$\text{span}_{\mathbb{C}}\{f_0, \dots, f_n\} \in \text{Gr}_{n+1} - Z,$$

let E_f be the rank n bundle on \mathbb{P}^n defined by f as in Proposition 5.1. We know that, $E_f \cong E_{f'}$ if and only if f differs from f' by an invertible linear transformation, i.e., they correspond to the same point in $\text{Gr}_{n+1} - Z$. In this way we get a continuous family $\{E_f \mid f \in \text{Gr}_{n+1}(V) - Z\}$ of pairwise non-isomorphic rank n algebraic vector bundles on \mathbb{P}^n . A simple dimension counting shows that the dimension of $\text{Gr}_{n+1}(V) - Z$ tends to infinity as a approaches infinity. By Theorem 5.7 and Lemma 5.6, $\{\pi^*(E_f) \mid f \in \text{Gr}_{n+1}(V) - Z\}$ are pairwise non-isomorphic algebraic vector bundles on X_n . Moreover, by Theorem 5.4 (1) each $\rho^*(E_f)$ is a trivial bundle on Y_n . Therefore, we finish the proof of the theorem. \square

Remark 5.9. *In the proof of Theorem 5.8, furthermore one can show that Z is irreducible and*

$$\dim Z = \text{Gr}_{n+1}(V) - 1.$$

6. A THEOREM OF SWAN

In this section, we present a proof for Theorem 3.2(2) communicated to the author by Professor Swan in an email.

Theorem 6.1 (Swan, [Sw]). *Write*

$$R = \mathbb{C}[x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n] / \langle x_0 y_0 + x_1 y_1 + \dots + x_n y_n - 1 \rangle.$$

Let $\{f_0, \dots, f_n\}$ be homogeneous polynomials of x_0, \dots, x_n with

$$\text{rad}(f_0, f_1, \dots, f_n) = (x_0, x_1, \dots, x_n).$$

If $n! \nmid \prod \deg f_i$, then $(f_0, f_1, \dots, f_n) \in R^{n+1}$ is not completable.

Proof. Let $i : S^{2n+1} \rightarrow \mathbb{C}^{n+1} - \{0\}$ and $r : \mathbb{C}^{n+1} - \{0\} \rightarrow S^{2n+1}$ be the natural inclusion and projection, respectively. They are homotopy inverse to each other. Let

$$f = (f_0, \dots, f_n) : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}^{n+1} - \{0\}$$

and

$$g = r \circ f \circ i.$$

By [ST], it suffices to show that the map $g : S^{2n+1} \rightarrow S^{2n+1}$ has degree $\prod \deg f_i$.

One knows that the homology group $H_{2n+1}(\mathbb{C}^{n+1} - \{0\}, \mathbb{Z})$ is isomorphic to \mathbb{Z} . We define the degree of the map f by the scalar of the multiplication of its induced action on $H_{2n+1}(\mathbb{C}^{n+1} - \{0\}, \mathbb{Z})$. It is clear that the degrees of f and g are equal. For maps from $\mathbb{C}^{n+1} - \{0\}$ to itself of the form $f = (f_0, \dots, f_n)$, it is clear that the degree is multiplicative for the composition of maps. After composing f with a map of the form $(z_0^{a_0}, \dots, z_n^{a_n})$, we may assume that f_i are of the same degree d .

Consider the projective space \mathbb{P}^{n+1} with homogeneous coordinates $[w : z_0 : \dots : z_n]$. Let $a \in S^{2n+1} \subset \mathbb{C}^{n+1}$ be a regular value of f . Define X by the set of points such that

$$f_i(z) - a_i w^d = 0, \quad i = 0, 1, \dots, n.$$

Then X is not empty by the dimension theorem. However the intersection of X with $\{[w : z_0 : \dots : z_n] \in \mathbb{C}^{n+1} \mid w = 0\}$ is empty by the unimodularity assumption. Therefore we have $\dim X = 0$. This means that X is a finite set. By Bezout's theorem, $\deg X = \prod \deg f_i$. Now X lies in $\{[w : z_0 : \dots : z_n] \in \mathbb{C}^{n+1} \mid w \neq 0\}$. Making $w = 1$, we get that X

is equal to $f^{-1}(a)$. As a is a regular value of f , all points of X will have multiplicity 1 and therefore X will have $\prod \deg f_i = d^{n+1}$ distinct points. It is clear that the radial projection r gives a bijection $f^{-1}(a) \rightarrow g^{-1}(a)$. Thus $g^{-1}(a)$ also has d^{n+1} distinct points.

Lemma 6.2. *If $a \in S^{2n+1}$ is a regular point of g , then the Jacobian of g at each point $p \in g^{-1}(a)$ is positive and a is also a regular value of f .*

Lemma 6.2 indicates that f has a regular value $a \in S^{2n+1}$. For such an a , Lemma 6.2 indicates that the number of points in $g^{-1}(a)$ is equal to the degree of g . By the above argument, the number of points in $g^{-1}(a)$ is equal to d^{n+1} . Therefore $\deg g = d^{n+1}$. \square

Proof of Lemma 6.2. Given a point $p \in g^{-1}(a)$, changing f to some $f'(z) = \lambda f(z)$ for some positive real number λ if necessary, we may assume that $f(p) = a$ (i.e., $p \in f^{-1}(a)$). The tangent spaces of $\mathbb{C}^{n+1} - \{0\}$ at p and a admit decompositions $T_p = N + S$, $T_a = N' + S'$, where N, N' are the 1-dimensional subspaces of normal vectors, and S, S' are the tangent spaces of S^{2n+1} at p, a . The tangent map $i_{*,p}$ is an injective map with image S , and the tangent map $r_{*,a}$ is a surjective map with kernel N' . Since f is homogeneous of degree d , we have $f_{*,p}(N) = N'$ and it is a positive scalar multiplication (here we identify N, N' using a non-zero normal vector field on $S^{2n+1} \subset \mathbb{C}^{n+1} - 0$). Thus $f_{*,p}$ is a triangular matrix with $f_{*,p}|_N$ and $g_{*,p}$ the block diagonal parts. The linear map $f_{*,p}|_N$ is clearly a positive scalar multiplication. Moreover, since f is holomorphic, we have $\det(f_{*,p}) \geq 0$. Hence $\det(g_{*,p}) \geq 0$. Therefore $\det(g_{*,p}) > 0$ since a is a regular value of g . By this we get $\det(f_{*,p}) > 0$. Similarly, we can show that $\det(f_{*,q}) > 0$ for any other $q \in f^{-1}(a)$. Therefore a is a regular value of f . \square

7. SOME REMARKS

Based on our study of algebraic vector bundles on \mathbb{P}^n with $e(E) = 1$ and their pull-backs to $X_n = \mathbb{C}^{n+1} \setminus \{0\}$ and $Y_n = \mathrm{SL}_{n+1} / \mathrm{SL}_n$, we ask the following questions.

Question 7.1. *Given an algebraic vector bundle E on \mathbb{P}^n ,*

- (1) *is ρ^*E a trivial bundle on X_n whenever $\mathrm{rank} E \geq n + 1$?*
- (2) *In the case of $e(E) = 1$, when is E stable or semi-stable?*
- (3) *In the case that E is non-split and has rank at most $n - 1$, can it satisfy that $M_i(E) = 0$ for $2 \leq i \leq n - 2$?*

A theorem of Kumar-Peterson-Rao (cf. [KPR]) confirms the non-existence in Question 7.1 (3) in the case that n is even. In the case that n is odd, it implies that the rank is at least $n - 1$.

Theorem 7.1 ([KPR]). *If E is a non-split algebraic vector bundle on \mathbb{P}^n with $M_i(E) = 0$ for $2 \leq i \leq n - 2$, then $\mathrm{rank} E \geq 2\lfloor \frac{n}{2} \rfloor$.*

The 3-term resolution we constructed for bundles E on \mathbb{P}^n with $M_i(E) = 0$ for $1 \leq i \leq n - 2$ is a special case of a general resolution theorem.

Proposition 7.2 (Three-term resolution). *Given an algebraic vector bundle E of rank r on $P = \mathbb{P}^n$, we have a (canonical) exact sequence of vector bundles*

$$0 \longrightarrow E' \longrightarrow E'' = \bigoplus_{1 \leq i \leq e+r} \mathcal{O}_{\mathbb{P}^n}(b_i) \longrightarrow E \longrightarrow 0$$

such that $e = e(E)$ and $H^0(E''(k)) \rightarrow H^0(E(k))$ is surjective for $k \in \mathbb{Z}$. For such an exact sequence, we have $M_1(E') = 0$ and $H^i(E'(k)) \cong H^{i-1}(E(k))$ for $2 \leq i \leq n - 1$ and $k \in \mathbb{Z}$.

By Proposition 7.2 and Theorem 2.1, we have the following resolution.

Proposition 7.3 (Syzygy resolution). *Given an algebraic vector bundle E of rank r on $P = \mathbb{P}^n$, there is a (canonical) exact sequence of vector bundles*

$$0 \longrightarrow E_n \xrightarrow{\phi_n} \cdots \longrightarrow E_1 \xrightarrow{\phi_1} E = E_0 \longrightarrow 0$$

such that each E_i ($i \geq 1$) is a direct sum of line bundles, $\text{rank ker } \phi_i = e(\text{Im } \phi_i)$ for $1 \leq i \leq n$, and $H^0(E_i(k)) \longrightarrow H^0(\text{Im } \phi_i(k))$ is surjective for $1 \leq i \leq n$ and $k \in \mathbb{Z}$.

Definition 7.4. *We call*

$$s(E) = \max\{i | E_i \neq 0\} - 1$$

the complexity of E .

Remark 7.5. *By Proposition 7.2, the complexity $s(E)$ is equal to*

$$\min\{i | M_j(E) = 0, \forall j, 1 \leq j \leq n - i\} - 1.$$

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